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ON CLASS (BD) OPERATORS OF ORDER (n+k)

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Abstract

In this paper, we introduce the class (BD) of order $(n + k)$ *operators acting on the classical Hilbert space* $\mathcal{H}$*. An operator* $T \in \mathcal{B}(\mathcal{H})$ *is said to belong to the class (BD) of order* $(n+k)$ *if* $T^{*2}(T^D)^2$ *commutes with* $(T^{*(n+k)}T^D)^2$ *, that is,* $\left[T^{*2}(T^D)^2,\ (T^{*(n+k)}T^D)^2\right] = 0$*. We investigate the properties of this class and analyze its relation to the* $(n + k)$*- power D-operator. This study explores various aspects such as unitary equivalence, restriction to closed subspaces, and the behavior of these operators under complex conjugation. In addition, we examine the connection between (BD) operators of order* $(n + k)$ *and D-operators of the same order.*

Keywords: *D-operator, Normal, N Quasi D-operator, complex symmetric operators, n-power D-operator, (BD) operators.*

Introduction

Throughout this paper, $\mathcal{H}$ denotes the usual Hilbert space over the complex field, and $\mathcal{B}(\mathcal{H})$ denotes the Banach algebra of all bounded linear operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$. The study of operators on Hilbert spaces has a rich history, with significant contributions in the development of various classes of operators. These operators are fundamental objects in functional analysis and have applications in quantum mechanics, signal processing, and other areas of mathematics and engineering.

A bounded linear operator T is said to be in class (Q) if $T^{*2}T^2 = (T^*T)^2$ [2]. This class was introduced to study operators with certain algebraic properties and has since been extended into other forms. For example, class (Q) includes operators for which $T^{*2}T^{2n} = (T^*T^n)^2$ [6]. Additionally, the n-power class (Q), the quasi-M class (Q), and the (α, β)-class (Q) have been studied to understand their distinct properties and applications [9].

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be in class (BQ) if it satisfies the relation $T^{*2}T^2(T^*T)^2 = (T^*T)^2T^{*2}T^2$. This class captures operators that satisfy more complex algebraic commutation relations. Further research into this class has revealed interesting insights into the structure of such operators and their applications in various fields [10].

Additionally, operators can be classified as D-operators if they satisfy the relation $T^{*2}(T^D)^2 = (T^{*(n+k)}T^D)^2$, where T^D is the Drazin inverse of T [1]. The Drazin inverse is a generalization of the Moore-Penrose inverse and plays a crucial role in the analysis of singular systems. The class of D-operators was introduced and studied by Abood in [1]. Results in [1] showed that D-operators preserve the scalar product and the unitary equivalence property, as well as the product of two D-operators; however, the sum of two D-operators is not necessarily a D-operator. On the other hand, the direct product and tensor product of two D-operators are also D-operators.

Beinane and Sid [5] made significant advancements in their study of operators on the Hilbert space

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$\mathcal{H}$. They introduced a variety of new operators, including the α-m-quasi- normal operators, denoted by $[\alpha(QN)_m]$, which exhibit unique properties that differentiate them from other operators studied previously. Furthermore, Beinane and Sid [5] extended their research by incorporating the concept of powers of n, leading to the introduction of $n\alpha$ -m-quasi-normal operators, symbolized as $[n\alpha(QN)_m]$. These operators are an essential addition to operator theory as they encompass the intricate behavior of operators raised to a power, and their relations with the α-m-quasi-normal operators.

One of the focal points of their study was the exploration of the Drazin inverse in relation to Drazin invertible operators. In Beinane and Sid [5], the authors examined how the Drazin inverse interacts with α-m-quasi-normal operators and $n\alpha$-m-quasi-normal operators, providing deeper insights into the structural and functional aspects of these operators.

Mohsen [3] later extended the study of D-operators to (n, D)-quasi operators. Results by Mohsen [3] showed that the sum and product of these classes hold. They also investigated the scalar and power properties of these classes and proved that the tensor product and direct product of (n, D)-quasi operators are again (n, D)-quasi operators. The Drazin inverse is a generalized inverse used to study singular systems and has applications in control theory and differential equations. This concept was further extended to N quasi D-operators by Wanjala and Nyongesa [8] . A bounded linear operator T is an N quasi D-operator if it satisfies the relation

$$T(T^{*2}(T^D)^2) = T(T^{*(n+k)}T^D)^2N$$

where N is another bounded linear operator [8]. These operators have been studied to understand their behavior and potential applications.

In this paper, we introduce the class (BD) of order $(n+k)$ operators and investigate their properties and relations to other operator classes. We aim to fill the gap in the literature by exploring the commutation properties and equivalence relations of these operators. This study extends the existing knowledge of operator theory by providing new insights into the structure and behavior of (BD) operators.

Preliminaries

In this section, we introduce some results and definitions from previous research which forms the basis of our study.

Definition 0.1 (7). Let $Q : \mathcal{H} \longrightarrow \mathcal{H}$ be a bounded operator. Then

1. Q is called self-adjoint if it satisfies $Q^*a = Qa$ for each $a \in \mathcal{H}$.
2. Q is called normal if it satisfies $Q^*Qa = QQ^*a$ for each $a \in \mathcal{H}$.
3. Q is called hyponormal if it satisfies $Q^*Qa < QQ^*a$ for each $a \in \mathcal{H}$.
4. Q is called quasinormal if it satisfies $QQ^*Qa < Q^*QQa$ for each $a \in \mathcal{H}$.

Lemma 0.1 (4). Let $Q, S : \mathcal{H} \longrightarrow \mathcal{H}$ be two Drazin-invertible operators. Then:

1. $(Q^*)^D = (Q^D)^*$,
2. $(Q^{-1}S^D)^D = Q^{-1}(S^D)^D$,
3. $(S^k)^D = (S^D)^k$, for $k = 1, 2, \ldots$,
4. If $QS = SQ$, then $(QS)^D = Q^DS^D$,
5. If $QS = SQ = 0$, then $(Q + S)^D = Q^D + S^D$.

Lemma 0.2 (1). Let T and S be two Drazin invertible operators. Then:

(a) $T + S$ is Drazin invertible.
(b) If $TS + ST$, then $(TS)^D = S^DT^D$.
(c) If T and S commute, then $(T + S)^D = T^D + S^D - T^DTS^D$.
(d) If T is Drazin invertible, then $(T^n)^D = (T^D)^n$ and $(T^D)^n = (T^n)^D$.
(e) If T and S are nilpotent operators, then T^D and S^D are also nilpotent.

Definition 0.2 (3). Let $T \in \mathcal{B}(\mathcal{H})$ be a Drazin operator. Then T is called an (n, α)-quasi operator if it satisfies the identity $T^{*2}(T^D)^{2n} = (T^*T^{Dn})^2$.

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Main Results

Definition 0.3. A bounded linear operator T is said to belong to class (BD) of order $n + k$ provided that $T^{*2(n+k)}(T^D)^2$ commutes with $(T^{*(n+k)}T^D)^2$, where T^D is the Drazin inverse of T.

Theorem 0.3. Let $T \in \mathcal{B}(\mathcal{H})$ be such that $T \in (BD)$ of order $n + k$. Then the following are also in (BD) of order $n + k$:

(i) λT for any real scalar λ,
(ii) any $S \in \mathcal{B}(\mathcal{H})$ that is unitarily equivalent to T,
(iii) the restriction $T|\mathcal{M}$ to any closed subspace $\mathcal{M} \subseteq \mathcal{H}$.

Proof.

(i) Trivial by scalar multiplication.
(ii) Let $S \in \mathcal{B}(\mathcal{H})$ be unitarily equivalent to T, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $S = U^*TU$ and $S^* = U^*T^*U$. Since $T \in$ (BD) of order $n + k$, we have

$$T^{*2(n+k)}(T^D)^2(T^{*(n+k)}T^D)^2 = (T^{*(n+k)}T^D)^2T^{*2(n+k)}(T^D)^2$$

Then

$$S^{*2(n+k)}(S^D)^2(S^{*(n+k)}S^D)^2 = U^*T^{*2(n+k)}UU^*(T^D)^2UU^*T^{*(n+k)}UU^*T^DU^2 = U^*T^{*2(n+k)}(T^D)^2(T^{*(n+k)}T^D)^2U,$$

and similarly,

$$(S^{*(n+k)}S^D)^2S^{*2(n+k)}(S^D)^2 = U^*T^{*(n+k)}UU^*T^DU^2U^*T^{*2(n+k)}UU^*(T^D)^2U = U^*(T^{*(n+k)}T^D)^2T^{*2(n+k)}(T^D)^2U,$$

Hence,

$S^{*2(n+k)}(S^D)^2(S^{*(n+k)}S^D)^2 = (S^{*(n+k)}S^D)^2S^{*2(n+k)}(S^D)^2$, and thus S $\in$ (BD) of order $n + k$.

(iii) If $T \in (BD)$ of order $n + k$, then by definition we have

$$T^{*2(n+k)}(T^D)^2(T^{*(n+k)}T^D)^2 = (T^{*(n+k)}T^D)^2T^{*2(n+k)}(T^D)^2$$

Let $\mathcal{M} \subseteq \mathcal{H}$ be a closed subspace such that $T\mathcal{M} \subseteq \mathcal{M}$, and let $T|\mathcal{M}$ denote the restriction of T to $\mathcal{M}$. Then the same commutation relation holds on M, implying that the restriction $T|\mathcal{M}$ $\in$ (BD) of order $n + k$.

Hence, we have:

$$(T/M)^{*2(n+k)}((T/M)^D)^2\left((T/M)^{*(n+k)}(T/M)^D\right)^2 = \left(T^{*2(n+k)}/M\right)((T^D)^2/M)\left(\left(T^{*(n+k)}/M\right)(T^D/M)\right)^2$$
$$= \left(\left(T^{*(n+k)}T^D\right)^2/M\right)\left(T^{*2(n+k)}(T^D)^2/M\right)$$
$$= \left(\left(T^{*(n+k)}/M\right)(T^D/M)\right)^2(T/M)^{*2(n+k)}((T/M)^D)^2$$

Therefore, $T/M \in (BD)$ of order $n + k$.

Theorem 0.4. If $T \in \mathcal{B}(\mathcal{H})$ is a D-operator of order $n + k$, then T $\in$ (BD) of order $n + k$.

Proof. Suppose T is a D-operator of order $n + k$, then by definition,

$$T^{*2(n+k)}(T^D)^2 = (T^{*(n+k)}T^D)^2$$

Post-multiplying both sides by $T^{*2(n+k)}(T^D)^2$, we obtain:

$$T^{*2(n+k)}(T^D)^2T^{*2(n+k)}(T^D)^2 = (T^{*(n+k)}T^D)^2T^{*2(n+k)}(T^D)^2$$

This simplifies to:

$$T^{*2(n+k)}(T^D)^2T^{*(n+k)}T^DT^{*(n+k)}T^D = (T^{*(n+k)}T^D)^2T^{*2(n+k)}(T^D)^2$$

Thus,

$$T^{*2(n+k)}(T^D)^2(T^{*(n+k)}T^D)^2 = (T^{*(n+k)}T^D)^2T^{*2(n+k)}(T^D)^2,$$

which shows that $T \in (BD)$ of order $n + k$.

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Theorem 0.5. Let $S \in (BD)$ of order $n + k$ and $T \in (BD)$ of order $n + k$. If both S and T are doubly commuting, then $ST \in (BD)$ of order $n + k$.

Proof. Since S and T are doubly commuting, we compute:

$$\begin{aligned}(ST)^{*2(n+k)}((ST)^D)^2((ST)^{*(n+k)}(ST)^D)^2 &= S^{*2(n+k)}T^{*2(n+k)}(S^D)^2(T^D)^2((ST)^{*(n+k)}(ST)^D)^2\\ &= S^{*2(n+k)}T^{*2(n+k)}(S^D)^2(T^D)^2\left(S^{*(n+k)}T^{*(n+k)}S^DT^D\right)^2\\ &= S^{*2(n+k)}T^{*2(n+k)}(S^D)^2(T^D)^2\left(S^{*(n+k)}S^DT^{*(n+k)}T^D\right)^2\\ &= S^{*2(n+k)}T^{*2(n+k)}(S^D)^2(T^D)^2\left(S^{*(n+k)}S^D\right)^2\left(T^{*(n+k)}T^D\right)^2\\ &= T^{*2(n+k)}(T^D)^2S^{*2(n+k)}(S^D)^2\left(S^{*(n+k)}S^D\right)^2\left(T^{*(n+k)}T^D\right)^2\end{aligned}$$

(Since $S \in (BD)$ of order $n + k$)

$$= \left(S^{*(n+k)}S^D\right)^2T^{*2(n+k)}(T^D)^2\left(T^{*(n+k)}T^D\right)^2S^{*2(n+k)}(S^D)^2$$

(Since $T \in (BD)$ of order $n + k$)

$$\begin{aligned}&= \left(S^{*(n+k)}S^D\right)^2\left(T^{*(n+k)}T^D\right)^2T^{*2(n+k)}(T^D)^2S^{*2(n+k)}(S^D)^2\\ &= \left(S^{*(n+k)}S^D\right)^2\left(T^{*(n+k)}T^D\right)^2S^{*2(n+k)}T^{*2(n+k)}(S^D)^2(T^D)^2\\ &= ((ST)^{*(n+k)}(ST)^D)^2(ST)^{*2(n+k)}((ST)^D)^2\end{aligned}$$

Hence, $ST \in (BD)$ of order $n + k$.

Theorem 0.6. Let $T \in \mathcal{B}(\mathcal{H})$ be an operator in class (BD) of order $n + k$, such that $T = CT^*C$, where C is a conjugation on $\mathcal{H}$. If C commutes with $T^{*2(n+k)}(T^D)^2$ and with $\left(T^{*(n+k)}T^D\right)^2$, then T is a D-operator of order $n + k$.

Proof. Let $T \in (BD)$ of order $n + k$ and assume T is complex symmetric, that is, $T = CT^*C$, where C is a conjugation. Then we have:

$$\begin{aligned}&T^{*2(n+k)}(T^D)^2\left(T^{*(n+k)}T^D\right)^2\\ &= \left(T^{*(n+k)}T^D\right)^2T^{*2(n+k)}(T^D)^2\end{aligned}$$

Since $T = CT^*C$, we use this property in the identity:

$$\begin{aligned}&T^{*2(n+k)}(T^D)^2\left(T^{*(n+k)}T^D\right)^2\\ &= T^{*2(n+k)}(T^D)^2CT^DCCT^{*(n+k)}CCT^DCCT^{*(n+k)}C\\ &= \left(T^{*(n+k)}T^D\right)^2CT^DT^{*(n+k)}T^DT^{*(n+k)}C\end{aligned}$$

Now using the commutation of C with both $T^{*2(n+k)}(T^D)^2$ and $\left(T^{*(n+k)}T^D\right)^2$, we get:

$$\begin{aligned}&T^{*2(n+k)}(T^D)^2C(T^D)^2T^{*2(n+k)}C\\ &= \left(T^{*(n+k)}T^D\right)^2C\left(T^{*(n+k)}T^D\right)^2C.\end{aligned}$$

Therefore,

$$\begin{aligned}&T^{*2(n+k)}(T^D)^2T^{*2(n+k)}(T^D)^2\\ &= \left(T^{*(n+k)}T^D\right)^2\left(T^{*(n+k)}T^D\right)^2,\end{aligned}$$

which implies

$$T^{*2(n+k)}(T^D)^2 = \left(T^{*(n+k)}T^D\right)^2$$

Hence, T is a D-operator of order $n + k$.

Conclusion and Recommendations

In this paper, we introduced the class (BD) of order $(n + k)$ operators acting on the classical Hilbert space $\mathcal{H}$ and investigated its properties and relations to other operator classes. We defined an operator $T \in \mathcal{B}(\mathcal{H})$ as belonging to class (BD) of order $n + k$ if $T^{*2}(T^D)^2$ commutes with $\left(T^{*(n+k)}T^D\right)^2$.

Through our analysis, we explored several key properties of these operators, including their

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unitary equivalence, behavior under restriction to closed subspaces, and behavior under complex conjugation.

Our results demonstrate that operators in the class (BD) of order $n+k$ exhibit intriguing algebraic and structural properties. We established that any operator T in class (BD) of order $n+k$ remains in this class under scalar multiplication by any real λ, under unitary equivalence, and when restricted to closed subspaces of $\mathcal{H}$. Additionally, we proved that if an operator $n+k$ is a D-operator of order $n+k$, then it belongs to the class (BD) of the same order.

We also explored the connection between doubly commuting (BD) operators of order $n+k$, showing that the product of such operators is also in class (BD) of order $n+k$. Furthermore, we established that if T is a complex symmetric operator in class (BD) of order $n+k$ and if the conjugation operator commutes with $T^{*2}(T^D)^2$ and $\left(T^{*(n+k)}T^D\right)^2$, then T is a D-operator of order $n+k$.

Overall, this study extends the existing knowledge of operator theory by providing new insights into the structure and behavior of (BD) operators of order $n+k$. The properties and relationships we have identified offer potential avenues for further research and applications in various fields of mathematics and engineering.

We recommend further studies to be done on class (BD) on fuzzy Hilbert space and Fuzzy soft Hilbert space.

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Credit authorship contribution statement

Both authors contributed equally to all aspects of this work, including conceptualization, writing, and final approval of the manuscript.

Declaration of conflict of interest

The authors declare no conflict of interest.

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