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ON K'D-OPERATOR

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Abstract

*In this paper, we introduce the class of k-*D-Operator. A bounded linear operator T is said to be a k-*D-Operator if $T^{*2k}(T^D)^2 = (T^{*k}T^D)^2$ for a positive integer k. We investigate the basic properties of this class and also show that this class is closed under strong operator topology. Methodology mainly involved the use of adjoint properties of bounded operator T. Results show that this class converges to the strong operator topology.*

Keywords: *D-Operator, *D-Operator, Class (Q), K*D-Operator.*

Introduction

Throughout this paper, we consider as a separable complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ as the Banach algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is classified as n-normal if $T^xT^n = T^nT^x$, where T^x denotes the adjoint of T. More specifically, an operator T is termed normal if $T^xT = TT^x$, and quasinormal if $T(T^xT) = (T^xT)T$.

The concept of D-operators has been explored in various studies. For instance, an operator T is defined as a D-operator if $T^{*2}(T^D)^2 = (T^*T^D)^2$ [1]. Furthermore, an operator belongs to class (Q) if $T^{*2}T^2 = (T^*T)^2$ [5]. This concept was extended to the n-power class (Q), where an operator T satisfies $T^{*2}(T^n)^2 = (T^*T^n)^2$ [9]. Similarly, an operator T is termed an n-D-operator if $T^{*2}(T^D)^{2n} = (T^*(T^D)^n)^2$ for any positive integer n. Notably, an n-D-operator reduces to a D-operator when $n = 1$.

A bounded linear operator T is said to belong to class (Q) if $T^{*2}T^2 = (T^*T)^2$ [5]. This class was introduced to explore operators with certain algebraic properties and has been expanded into other forms. For instance, class (Q) includes operators where $T^{*2}T^{2n} = (T^*T^n)^2$ for some positive integer n [8]. Moreover, the n-power class (Q), quasi M class (Q), and (α, β)-class (Q) have been investigated to comprehend their unique properties and applications [18,12,13].

Additionally, operators can be categorized as D-operators if $T^{*2}(T^D)^2 = (T^{*(n+k)}T^D)^2$, where T^D is the Drazin inverse of T [1]. The Drazin inverse is a generalization of the Moore-Penrose inverse and plays a crucial role in the analysis of singular systems. The class of D-operators was introduced and covered by Abood in [1]. Results in [1] demonstrated that D-operators preserve the scalar product, the unitary equivalence property, and the product property. However, the sum of two D-operators is not generally a D-operator, but the direct product and tensor product are also D-operators.

Beinane and Sid [2] in 2020 made significant strides in their study of operators on Hilbert space $\mathcal{H}$. Building upon their previous work, they introduced a variety of new operators, including the α-m-quasi-normal operators. These operators are denoted as $[\alpha(QN)^m]$ and exhibit unique $\mathcal{H}$ properties that distinguish

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them from other previously studied operators. Furthermore, Ben et al. [2] extended their research by incorporating the concept of power n, leading to the introduction of $n\alpha$-m-quasi-normal operators. These operators, symbolized as $[n\alpha(QN)^m]$, are a vital addition to the theory of operators as they encompass the complex behavior of operators raised to a power and their relationships with the a-m-quasi-normal operators.

Mohsen in [7] extended the study of D-operators to (n, D)-quasi operators. Results in [7] showed that the sum and product of these classes hold. They also investigated the scalar and power properties of these classes and proved the tensor product and direct product of (n, D)-quasi operators. The Drazin inverse, a generalized inverse used to study singular systems, has applications in control theory and differential equations. This concept was further extended to N Quasi D-operators by Wanjala and Nyongesa in [11]. A bounded linear operator T is called an N Quasi D-operator if

$$2T(T^{*2}(T^D)^2) = N\left(T^{*(n+k)}T^D\right)^2 T$$

where N is another bounded linear operator [11]. These operators have been studied to understand their behavior and potential applications.

This paper aims to delve deeper into the properties and applications of these operators, building upon the foundational work by various authors. Abood and Kadhim [1] investigated the properties of D-operators, highlighting their unique characteristics and potential applications in various mathematical contexts. Ben-Israel and Greville [2] , as well as Campbell and Meyer [3], provided extensive insights into generalized inverses and their applications, forming a crucial theoretical foundation for understanding the behavior of D-operators.

Dana and Yousef [4] explored classes of D-normal and D-quasinormal operators on Hilbert spaces, contributing to a broader understanding of these operators' structural properties. Jibril [5] examined operators satisfying the condition

$$T^{*2}T^2 = (T^*T)^2$$

while Panayappan and sivamani [9] extended this concept to n-power class (Q) operators. By synthesizing these contributions, this paper seeks to provide a comprehensive overview of D-operators and their extensions, offering new perspectives and potential avenues for future research.

Preliminaries

Definition 0.1 ([10]). Let $‡ : \mathcal{H} \to \mathcal{H}$ be a bounded operator.

1. $‡$ is self-adjoint if $‡^* x = ‡ x$ for all $x \in \mathcal{H}$.
2. $‡$ is normal if $‡^*‡ = ‡‡^*$.
3. $‡$ is hyponormal if $‡^*‡ < ‡‡^*$.
4. $‡$ is quasinormal if $‡‡^*‡ = ‡^*‡‡$.

Lemma 0.1 ([10], [6]). Let $‡, S : \mathcal{H} \to \mathcal{H}$ be two Drazin-invertible operators:

1. $(‡^*)^D = (‡^D)^*$
2. $(‡^{-1} S^D)^D = ‡^{-1} (S^D)^D$
3. $(S^k)^D = (S^D)^k$ for $k = 1, 2, \ldots$
4. If $‡ S = S ‡$, then $(‡ S)^D = ‡^D S^D$
5. If $‡ S = S ‡ = 0$, then $(‡ + S)^D = ‡^D + S^D$

Lemma 0.2. Let T and S be two Drazin-invertible operators, then:

(a) $T + S$ is Drazin-invertible.
(b) $(TS)^D = S^D T^D$ if $TS = ST$.
(c) If T and S commute, then $(T + S)^D = T^D + S^D - T^D S^D T^D S^D T^D$.
(d) If T is Drazin-invertible, then $(T^n)^D = (T^D)^n$ and $(T^D)^n = (T^n)^D$.
(e) If T and S are nilpotent operators, then T^D and S^D are also nilpotent.

Definition 0.2 ([7]). Let $T \in \mathcal{B}(\mathcal{H})$ be a Drazin-invertible operator. Then T is called an

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(n, α)-quasi operator if it satisfies

$$T^{*2}(T^D)^{2n} = \left(T^*(T^{Dn})\right)^2$$

Main Results

Definition 0.3. Let $T \in \mathcal{B}(\mathcal{H})$ be a Drazin-invertible operator. Then T is called a k-*D-operator, denoted by $[k^*D]$, if it satisfies

$$T^{*2k}(T^D)^2 = (T^{*k}T^D)^2$$

for any positive integer k.

Proposition 0.3. Let $T \in [k^*D]$, then the following properties hold:

(i) $\lambda T \in [k^*D]$ for every scalar λ.
(ii) $S \in [k^*D]$ for every $S \in \mathcal{B}(\mathcal{H})$ that is unitarily equivalent to T.
(iii) The restriction $T|_M$ of T to any closed subspace M of $\mathcal{H}$ which reduces T is in $[k^*D]$.
(iv) $T^D \in [k^*D]$.

Proof.

(i) Trivial by scalar multiplication.
(ii) Since S is unitarily equivalent to T, there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $S = UTU^*$

Hence,

$$S^{*2k}(S^D)^2 = (UT^{*k}U^*)^2(UT^DU^*)^2$$
$$= (UT^{*k}U^*)(UT^{*k}U^*)(UT^DU^*)(UT^DU^*)$$
$$= UT^{*k}T^{*k}T^DT^DU^* = UT^{*2k}(T^D)^2U^*$$
$$= U(T^{*k}T^D)^2U^* = UT^{*k}T^DT^{*k}T^DU^*$$
$$= (UT^{*k}U^*)(UT^DU^*)(UT^{*k}U^*)(UT^DU^*$$
$$= S^{*k}S^DS^{*k}S^D$$
$$= (S^{*k}S^D)^2$$

Thus, $S \in [k^*D]$.

(iii) For the restriction $T|M$ of T to a closed subspace $M \subseteq \mathcal{H}$ which reduces T, we have:

$$(T|M)^{*2k}((T|M)^D)^2$$
$$= (T|M)^{*k}(T|M)^{*k}(T|M)^D(T|M)^D$$
$$= (T^{*k}|M)(T^{*k}|M)(T^D|M)(T^D|M)$$
$$= (T^{*k}T^{*k}|M)(T^DT^D|M)$$
$$= (T^{*2k}|M)((T^D)^2|M) = (T^{*2k}(T^D)^2)|M$$
$$= (T^{*k}T^DT^{*k}T^D)|M$$
$$= ((T^{*k}T^D)|M)((T^{*k}T^D)|M)$$
$$= (T^{*k}|M)(T^D|M)(T^{*k}|M)(T^D|M)$$
$$= ((T|M)^{*k}(T|M)^D)^2$$

Hence, $T|M \in [k^*D]$.

(iv) Suppose $T \in [k^*D]$, then

$$T^{*2k}(T^D)^2 = (T^{*k}T^D)^2$$

Hence,

$$T^{*k}T^{*k}T^DT^D = T^{*k}T^DT^{*k}T^D$$

Taking adjoints on both sides:

$$(T^*)^D(T^*)^DT^kT^k = (T^*)^DT^k(T^*)^DT^k$$

Thus,

$$((T^D)^*)^2T^{2k} = ((T^D)^*T^k)^2$$

Hence,

$T^D \in [k^*D$

Proposition 0.4. The set of all k-*D-Operators is closed in the strong operator topology.

Proof. Let $\{T_q\}$ be a sequence of $[k^*D]$-operators with $T_q \to T$. We have to show that $T \in [k^*D]$.

Now $T_q \longrightarrow T$ implies that

$T_q^{*k} \to T^{*k}$ and $T_q^D \to T^D$

Thus,

$T_q^{*k}T_q^D \to T^{*k}T^D$ gives

$$\left(T_q^{*k}T_q^D\right)^2 \to (T^{*k}T^D)^2 \qquad 0.1$$

Similarly,

$$T_q^{*2k} \to T^{*2k} \text{ and } \left(T_q^D\right)^2 \to (T^D)^2$$

thus

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$$T_q^{*2k}\left(T_q^D\right)^2 \to T^{*2k}(T^D)^2 \qquad 0.2$$

Hence, from (0.1) and (0.2), we have:

$$||T^{*2k}(T^D)^2 - (T^{*k}T^D)^2|| = T^{*2k}(T^D)^2 - T_q^{*2k}\left(T_q^D\right)^2 + T_q^{*2k}\left(T_q^D\right)^2 - T^{*k}(T^D)^2||$$

$$\leq ||T^{*2k}(T^D)^2 - T_q^{*2k}\left(T_q^D\right)^2|| + ||T_q^{*2k}\left(T_q^D\right)^2 - T^{*k}(T^D)^2||$$

$$= ||T^{*2k}(T^D)^2 - T_q^{*2k}\left(T_q^D\right)^2|| + ||T_q^{*2k}\left(T_q^D\right)^2 - T^{*k}(T^D)^2||$$

$$\to 0 \quad as \quad q \to \infty$$

Thus,

$$T^{*2k}(T^D)^2 = (T^{*k}T^D)^2,$$

hence $T \in [k^*D]$.

Conclusion and Recommendations

This paper introduced the class of k-*D-operators acting on Hilbert space and studied its properties. We proved closure under scalar multiplication, unitary equivalence, and restriction. We also established the class is closed under strong operator topology.

These results extend current knowledge of operator classes and open pathways for further research in operator theory. We recommend this study to be explored in the context of Fuzzy soft and Fuzzy Hilbert spaces.

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Credit authorship contribution statement

Both authors contributed equally to all aspects of this work, including conceptualization, writing, and final approval of the manuscript.

Declaration of conflict of interest

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